Transport in weighted networks: superhighways and roads

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Wu, Braunstein, Havlin, Stanley, PRL (2006)
What is the research question?

• In complex network, different nodes or links have different importance in the transport process.

• How to identify the “superhighways”, the subset of the most important links or nodes for transport?

• Identifying the superhighways and increasing their capacity enables to improve transport significantly.
Networks with weights, such as “cost”, “time”, “bandwidth” etc. associated with links or nodes.

Many real networks such as world-wide airport network (WAN), *E Coli* metabolic network etc. are weighted networks.

Many dynamic processes are carried on weighted networks.

Minimum spanning tree (MST)

- The tree which connects all nodes with minimum total weight.
- Union of all “strong disorder” optimal paths between any two nodes.
- The MST is the part of the network that most of the traffic goes through.
- MST -- widely used in optimal traffic flow, design and operation of communication networks.

In strong disorder the weight of the path is determined by the largest weight along the path!
Optimal path – strong disorder
Random Graphs and Watts Strogatz Networks

\[ l_{\text{opt}} \sim N^{\frac{1}{3}} \]

N – total number of nodes

Analytically and Numerically

LARGE WORLD!!

Compared to the diameter or average shortest path or weak disorder

\[ l_{\text{min}} \sim \log N \]

(small world)

\[ n_0 \] - typical range of neighborhood without long range links

\[ \frac{N}{n_0} \] - typical number of nodes with long range links

Braunstein, Buldyrev, Cohen, Havlin, Stanley, Phys. Rev. Lett. 91, 247901 (2003);
Betweeness Centrality of MST

- Number of times a node (or link) is used by the set of all shortest paths between all pairs of node.
- Measure the frequency of a node being used by traffic.

\[ P_{\text{MST}} (C) \sim C^{-\delta_{\text{MST}}} \quad \delta_{\text{MST}} \approx 2 \]

For ER, scale free and real world networks

Minimum spanning tree (MST)

High centrality nodes
Incipient percolation cluster (IIC)

• IIC is defined as the largest component at percolation criticality.

• For a random scale-free or Erdös-Rényi graph, to get the IIC, we remove the links in descending order of the weight, until 

\[ \kappa \equiv \langle k^2 \rangle / \langle k \rangle \]

is < 2. At \( \kappa = 2 \), the system is at criticality. Then the largest connected component of the remaining structure is the IIC.

• The IIC can be shown to be a subset of the MST

MST and IIC

Superhighways and Roads

The IIC is a subset of the MST

Superhighways
Superhighways (SHW) and Roads
Mean Centrality in SHW and Roads

(a) ER Networks $\langle k \rangle = 4$

(b) SF Networks $\lambda = 4.5$

(c) SF Networks $\lambda = 3.5$

(d) 90x90 square lattice
The average fraction of pairs of nodes using the IIC

\[ <f> \sim g(\ell_{\text{MST}} / N^{\nu_{\text{opt}}}) \]

\[ \nu_{\text{opt}} = \begin{cases} 
\frac{\lambda - 3}{\lambda - 1} & 3 < \lambda < 4 \\
1/3 & \lambda > 4 \text{ and ER}
\end{cases} \]
How much of the IIC is used?

The IIC is only a ZERO fraction of the network of order $N^{2/3}$!!
Distribution of Centrality in MST and IIC

(a) ER Networks
\(<k>=4\)

(b) SF Networks
\(\lambda=4.5\)

(c) SF Networks
\(\lambda=3.5\)

(d) Square lattice

\(\lambda=3.5\)

\(\lambda=4.5\)

\(\lambda=3.5\)

\(\lambda=4.5\)
Theory for Centrality Distribution

For IIC inside the MST:

\[ n_\ell \propto \ell^3 \] for network at criticality

\[ n_\ell \] is number of nodes in MST within \( \ell \)

\[ S_\ell \sim \ell^2 \] for nodes in the IIC

Thus the number of nodes with centrality larger than \( n_\ell \) is

\[ m(C > n_\ell) \propto n_\ell^{1/3} \frac{S}{S_\ell} \approx \frac{n_\ell^{1/3}}{n_\ell^{2/3}} S \sim n_\ell^{-1/3} \]

for all \( \ell \) due to self-similarity. Thus,

\[ p_{IIC}(C) \propto C^{-4/3} \]

For the MST:

\[ m(C > n_\ell) \sim n_\ell^{1/3} \frac{N}{n_\ell} \approx \frac{n_\ell^{1/3}}{n_\ell} N \sim n_\ell^{-2/3} \] Thus,

\[ p_{MST}(C) = \frac{dm}{dn_\ell} \sim n_\ell^{-5/3} \sim C^{-5/3} \]

Good agreement with simulations!
Application: improve flow in the network

Comparison between two strategies:

sI: improving capacity of all IIC links--highways
sII: improving the highest centrality links in MST (same number as sI).

Assume: multiple sources and sinks: randomly choose $n$ pairs of nodes as sources and other $n$ nodes as sinks

We study two transport problems:

• Current flow in random resistor networks, where each link of the network represents a resistor. (Total flow, $F$: total current or conductance)

• Maximum flow problem from computer science, where each link of the network has an upper bound capacity. (Total flow, $F$: maximum possible flow into network)

Result: sI is better
Application: compare two strategies

**current flow and maximum flow**

Two types of transport

- **Current flow**: improve the conductance
- **Maximum flow**: improve the capacity

$sI$: improve the IIC links.

$sII$: improve the high $C$ links in MST.

$F_0$: flow of original network.

$F_{sI}$: flow after using $sI$.

$F_{sII}$: flow after using $sII$.

$N=2048$, $<k>=4$
Summary

• MST can be partitioned into superhighways which carry most of the traffic and roads with less traffic.

• We identify the superhighways as the largest percolation cluster at criticality -- IIC.

• Increasing the capacity of the superhighways enables to improve transport significantly. The superhighways of order $N^{2/3}$ -- a zero fraction of the network!!